# Calculating $\pi$ by programming and by hand

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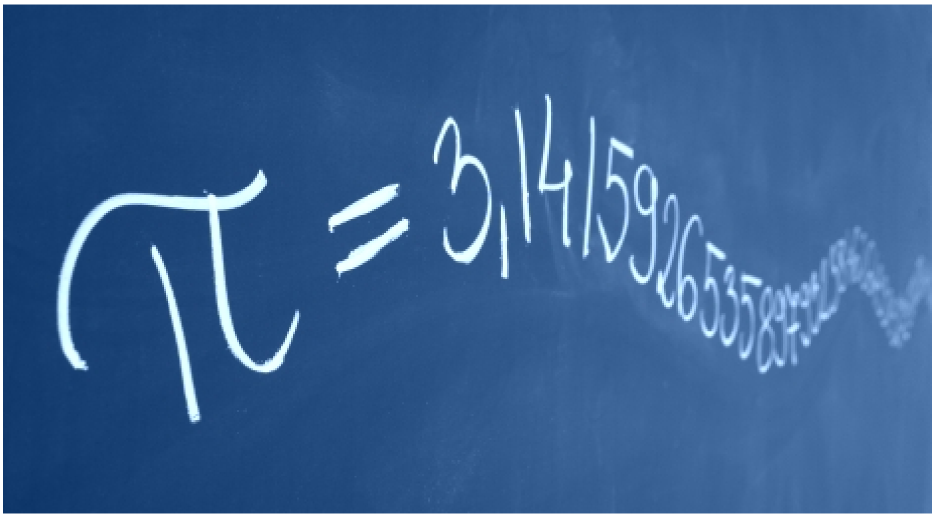


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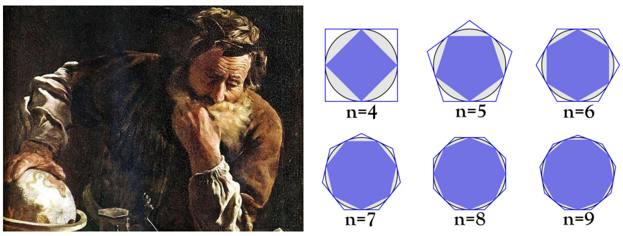
## Introduction

In this paper the focus is set on two different approximations of $\pi$, namely, the approach of Euler and of Machin. I will show the mathematics in which they are based on. Furthermore, I will write a program in R that successfully calculate the first thousands digits of $\pi$. Afterwards, I will use Machin's approach to calculate the first digits of $\pi$ by hand.

## History

The number $\pi$ has been known for thousands of years and it is even mentioned in the *Bible* which state that $\pi = 3.0$. A verse from the Bible reads *And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it about. (I Kings 7, 23).* The ancient Babylonians stated that $\pi \approx 3.12$ about 1900 BC[1] and the old Egyptian text *Rhind Papyrus*, from about 1650 BC, states that $\pi \approx 3.16$[2]. The first theoretical calculation of $\pi$ was done by Archimedes (287–212 BC)[3]. He approximated the area of a circle with the areas of two regular polygons: the polygon inscribed within the circle and the polygon within which the circle was circumscribed. Since the actual area of the circle lies between the areas of the inscribed and circumscribed polygons, the areas of the polygons gave upper and lower bounds for the area of the circle. From this approach he concluded that $\frac{223}{71} < \pi < \frac{22}{7}$ and the first 3 digits of $\pi$ were known. Note that the old Pyramids at Giza were build around 2560BC[4]. They knew at least 4 digits of $\pi$ and why this knowledge was lost is a mystery. It may imply the existence of a lost civilisation.



Picture of Archimedes and his approach.[2]

A similar approach was used by the Chinese mathematician Zu Chongzhi (429–501 AD). He would not have been familiar with Archimede's work, however, because his book has been lost we know little about his work. We know that he stated that $\pi$ is close to 355/113.[3]

## Approximations of $\pi$ using basic calculus

The next breakthrough came after the renaissance when Gottfried Leibniz (1646-1716) and Isaac Newton (1642-1727) formalised calculus as we know it today. In this time many series were produced which converted to $\pi$ or to a simple multiple of $\pi$. Probably the most famous is the one Leibniz discovered in 1674[3]:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ...$$

Another important approximation of $\pi$ was derived by John Machin in 1706 which he used to calculate the first 100 digits of $\pi$. He stated that $\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$. Let's take a closer look at them. Leibniz proved it in a different way. Nevertheless a nice proof of his series is shown below.

**Theorem 1.** *Leibniz's identity* $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ...$

*Proof.* Consider

$$\arctan(z) = \int_0^z \frac{1}{1+x^2}dx$$
$$= \int_0^z (1 - x^2 + x^4 - x^6 + x^8 - ...)dx$$
$$= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9} - ...$$

which has radius of convergence 1 and it does converge to $\arctan(z)$ at 1. Abel's theorem states that if the function $\zeta$ is defined as the power series $\sum_{n=1}^{\infty} a_n x^n$ on the interval $(-1, 1)$ and the series $\sum_{n=1}^{\infty} a_n$ converges to a number $A$, then the limit $\lim\limits_{x\to 1} \zeta(x)$ exist and is equal to $A$. Note that in this particular case we know the limit of $\zeta(x) = \arctan(x)$ exists, and what we care about is the equality. The series obviously converges at 1, and so by Abel's theorem it represents $\arctan(z)$ here. $\square$

The general series above is known as Gregory's series and letting $z = 1$, we get Leibniz's series which is very beautiful, but for computational purposes it converges very slowly. It would take several hundred terms just to get 2-digit accuracy. In fact, calculating $\pi$ to 10 correct decimal places requires about five billion terms. However, the Leibniz formula can be used to calculate $\pi$ to high precision using an elegant convergence acceleration technique, called the *Euler transformation.* We will get back to that later.

**Theorem 2.** *Machin's identity:* $\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$.

*Proof.* Recall the trigonometric identities

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

So for $\theta \notin \left\{2\pi, \frac{3}{2}\pi, \frac{\pi}{2}, 0\right\}$ we have that

$$\tan(2\theta) = \frac{2\cos(\theta)\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)} = \frac{\frac{2\cos(\theta)\sin(\theta)}{\cos^2(\theta)}}{\frac{\cos^2(\theta) - \sin^2(\theta)}{\cos^2(\theta)}} = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}.$$

Let $\theta = \arctan(\frac{1}{5})$, thus

$$\tan(2\theta) = \frac{2 \cdot \frac{1}{5}}{1 - \frac{1}{25}} = \frac{5}{12}$$

and

$$\tan(4\theta) = \frac{2\tan(2\theta)}{1 - \tan^2(2\theta)} = \frac{2 \cdot \frac{5}{12}}{1 - (\frac{5}{12})^2} = \frac{120}{119} = 1 + \frac{1}{119}.$$

Recall that for angles $\alpha$ and $\beta$ we have that

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}.$$

From this we get

$$\tan(4\theta - \frac{\pi}{4}) = \frac{\tan(4\theta) - 1}{1 + \tan(4\theta)} = \frac{1 + \frac{1}{119} - 1}{1 + 1 + \frac{1}{119}} = \frac{1}{119 + 119 + 1} = \frac{1}{239}.$$

Taking arctan of both sides of this we get

$$4\theta - \frac{\pi}{4} = \arctan(\frac{1}{239}) \Rightarrow \frac{\pi}{4} = 4\theta - \arctan(\frac{1}{239}) = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}).$$

$\square$

This series converges more quickly and it was derived by John Machin. He proved it the very same way as above. As mentioned, he used it to calculate 100 digits of $\pi$ in 1706. Gregory's series allows this to be evaluated reasonably efficiently: the first term works well with decimal arithmetic, and the second converges rapidly.

Picture of John Machin.[5]

The approximation in raw form is shown below.

$$\pi = 16 \cdot (\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - ...) - 4 \cdot (\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - ...)$$

This was the approximation used by $\pi$-digit hunters until the late 20th century. The British amateur mathematician William Shanks used the approach to calculate the first 707 digits of $\pi$ and it took him about 15 years. It was accomplished in 1873, however, it was only correct up to the 527 places. Shanks spend his time on calculating mathematical constants. He would calculate new digits all morning, and then he would spend all afternoon checking his morning's work. In 1947 D. F. Ferguson used the same approach to calculate the first 808 digits using a mechanical desk calculator. Since then more digits of $\pi$ where calculated due to the development of computers. In 1976 a better method was available, due independently to Brent and Salamin. We will get back to that later.

## Euler's transformation

The Swiss mathematician Leonhard Euler (1707-1783) developed an elegant convergence acceleration technique called the *Euler transformation*. We have already noted that Leibniz's series converges very slowly and is quite useless for any practical purpose. Nevertheless, we can ask if there is some transformation which can be applied to the series that would generate an equivalent series that converge faster. This is the motivation. The simplest thing to try is to notice that since this is an alternating series with terms of diminishing absolute value, we can take the terms in pairs and write this series as

$$\frac{\pi}{4} = (1 - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{7}) + (\frac{1}{9} - \frac{1}{11})...$$

$$= \frac{2}{3} + \frac{2}{5 \cdot 7} + \frac{2}{9 \cdot 11} + ... = 2\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)}.$$

Clearly this series is more convenient for computation than Leibniz's series because it simply avoids the computational difficulty of adding and subtracting numbers that mainly cancel out. However, this series doesn't actually converge any faster than Leibniz's series, since each term is just the sum of two consecutive terms and after $n$ terms we are still only within a distance of about $1/n$ from the sum. The $n$th term of the new series is $O(\frac{1}{n^2})$ and is very similar to Euler's famous formula $\frac{\pi^2}{6} = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ which he proved in 1734, also known as the *Basel Problem*. Euler found a method of transforming series exactly like this and his new series improved the rate of convergence significantly. He came up with a general method that works for a large class of alternating series.

Picture of Leonhard Euler.[6]

Let's set the notation. For any sequence of real numbers $(x_0, x_1, ...)$, we define the *forward* difference $\triangle x_0, \triangle x_1, ...$ by $\triangle x_k = x_{k+1} - x_k$ and we define *backward* difference sequence $\bigtriangledown x_k = x_k - x_{k-1} = - \triangle x_k$. Euler's transformation can be written in terms of either $\triangle$ or $\bigtriangledown$. We will use $\bigtriangledown$. Note that we can iterate this process: we can take differences of differences and so on. With the convention $\bigtriangledown^0 x_k = x_k$ and $\bigtriangledown = \bigtriangledown^1$ we have that

$\bigtriangledown^0 x_0 = x_0$

$\bigtriangledown^0 x_1 = x_1$

$\bigtriangledown^0 x_2 = x_2$

$\bigtriangledown^0 x_3 = x_3$

$\bigtriangledown^1 x_0 = x_0 - x_1$

$\bigtriangledown^1 x_1 = x_1 - x_2$

$\bigtriangledown^1 x_2 = x_2 - x_3$

$\bigtriangledown^2 x_0 = \bigtriangledown x_0 - \bigtriangledown x_1 = x_0 - 2x_1 + x_2$

$\bigtriangledown^2 x_1 = \bigtriangledown x_1 - \bigtriangledown x_2 = x_1 - 2x_2 + x_3$

$\bigtriangledown^3 x_0 = \bigtriangledown^2 x_0 - \bigtriangledown^2 x_1 = x_0 - 3x_1 + 3x_2 - x_3$

We see from induction that

$$\bigtriangledown^k a_m = \sum_{j=0}^{k} (-1)^j \binom{k}{j} x_j.$$

With this notation, the Euler Transform of an alternating series $\sum\limits_{j=0}^{\infty} (-1)^j a_j$ is produced by the following series of steps:

$$S = a_0 - a_1 + a_2 - a_3 = \frac{1}{2} a_0 + \frac{1}{2}((a_0 - a_1) - (a_1 - a_2) + (a_2 - a_3) - (a_3 - a_4) + ...)$$

$$= \frac{1}{2} a_0 + \frac{1}{4}(a_0 - a_1) + ((a_0 - 2a_1 + a_2) - (a_1 - 2a_2 + a_3) + (a_2 - 2a_3 + a_4) - ...).$$

So we replace the series $\sum_{j=0}^{\infty} (-1)^j a_j$ with the series

$$\sum_{k=0}^{\infty} \frac{\bigtriangledown^k a_0}{2^{k+1}}$$

or equivalently

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_j.$$

Note that the series $S$ itself is alternating, but each $a_k \geq 0$. For a large class of cases we have that the Euler transformation converge, and it converge to the same sum as the original series and finally this convergence is much faster. We will later use the Euler transformation on Leibniz's series and use it to get a much faster convergence, ie a series that converges to the same value as Leibniz series, but this new series convergence much faster to the same result. Euler did not prove any general theorems about this transformation. He did use it in some cases where he could show that it did converge to the original sum and converged much more quickly. It was formal proven by Felix Hausdorff (1868-1942) who showed that this method can be used to arrive at Euler's transformation. Later it has been simplified by Henri Cohen[7]. A nice detailed proof of this is shown below.

**Theorem 3.** *Euler's transformation converges quickly to the original sum.*

*Proof.* Assume that there is a positive weight function $w(t)$ such that the numbers $a_k$ are the moments of w:

$$a_k = \int_0^1 t^k w(t)dt.$$

We will use this to write the sum of series

$$\begin{aligned} S &= \sum_{k=0}^{\infty}(-1)^k a_k = \sum_{k=0}^{\infty}(-1)^k t^k w(t)dt \\ &= \int_0^1 \left( \sum_{k=0}^{\infty}(-1)^k t^k \right) w(t)dt = \int_0^1 \frac{1}{1+t} w(t)dt. \end{aligned}$$

Since $t \in (0,1)$ and it follow since the sum of a geometric series is $\sum_0^{\infty}(-1)^k t^k = \frac{1}{1+t}$. Now suppose that we have a sequence $\{P_n\}$ of polynomials where $P_n$ has degree $n$, so we can write

$$P_n(t) = \sum_{j=0}^{n} c_j^{(n)} t^j$$

and where $P_n \neq 0$. Define a function $\varphi$ as $\varphi = P_n - P(-1)$, we see that $\varphi = 0$ for $t = -1$ and from the basic factor theorem we know that $\varphi$ is divisible by $t-(-1)$ and thereby $t+1$.

Let us set

$$S_n = \frac{1}{P_n(-1)} \int_0^1 \frac{P_n(-1) - P_n(t)}{1+t} w(t)dt.$$

We can split this sum into two parts:

$$S_n = \int_0^1 \frac{1}{1+t} w(t)dt - \frac{1}{P_n(-1)} \int_0^1 \frac{P_n(t)}{1+t}.$$

As we have seen, the first term on the right hand is just hte sum $S$ of the series. We want to find a sequence of polynomials such that the second integral on the right is very small. For such a sequence of polynomials, $S_n$ will be very close to $S$. If we can do this right, $S_n$ will be easy to express, and will converge to $S$ faster than the original series does.

$$\begin{aligned} S_n &= S - \frac{1}{P_n(-1)} \int_0^1 \frac{P_n(t)}{1+t} w(t)dt \\ &= S - \frac{1}{P_n(-1)} \int_0^1 \left( \sum_{j=0}^{n} c_j^n t^j \right) \left( \sum_{k=0}^{\infty} (-t)^k \right) w(t)dt \\ &= S - \frac{1}{P_n(-1)} \int_0^1 \left( \sum_{\kappa=0}^{n} t^\kappa \sum_{j=0}^{\kappa} c_j^{(n)} (-1)^{\kappa-j} + \sum_{\kappa=n+1}^{\infty} t^\kappa \sum_{j=0}^{n} c_j^{(n)} (-1)^{\kappa-j} \right) w(t)dt \end{aligned}$$

where $\kappa := k + j$.

Now we are ready to pick the polynomials $P_n$. We will use $P_n = (1-t)^n$, since $P_n(-1) = 2^n$ grows rapidly and it's bounded by 1 on the interval $[0, 1]$, so the integral is bounded

$$\frac{1}{P_n(-1)} \int_0^1 \frac{P_n(t)}{1+t} w(t)dt = O(2^{-n})$$

for the right chose of $w(t)$ and we see that it goes to 0 quickly, ie $S_n \longrightarrow S$ quickly. We will now express $S_n$ more clearly. From the binomial theorem we have that

$$P_n(t) = (1-t)^n = \sum_{j=0}^{n} \binom{n}{j} (-t)^j.$$

and therefore

$$c_j^{(n)} = (-1)^j \binom{n}{j}.$$

With this we will continue our computation of $S_n$.

$$S_n = S - \frac{1}{2^n} \int_0^1 \sum_{k=0}^{n} (-1)^k t^k \sum_{j=0}^{k} \binom{n}{j} w(t)dt + \frac{1}{2^n} \int_0^1 \sum_{k=n+1}^{\infty} (-1)^k t^k \sum_{j=0}^{n} \binom{n}{j} w(t)dt.$$

We know that $\sum_{j=0}^{n} \binom{n}{j} w(t)dt = 2^n$, so the third term on the right is just

$$\frac{1}{2^n} \int_0^1 \sum_{k=n+1}^{\infty} (-1)^k t^k \cdot 2^n w(t)dt = \sum_{k=n+1}^{\infty} (-1)^k a_k$$

and so $S$ minus this term is just $\sum_{k=0}^{n}(-1)^k a_k$. Thus we have

$$\begin{aligned}
S_n &= \sum_{k=0}^{n}(-1)^k a_k - \frac{1}{2^n}\int_0^1 \sum_{k=0}^{n}(-1)^k t^k \sum_{j=0}^{k}\binom{n}{j} w(t)dt \\
&= \sum_{k=0}^{n}(-1)^k a_k \left(1 - \frac{\int_0^1 \sum_{k=0}^{n}(-1)^k t^k w(t)dt}{\sum_{k=0}^{n}(-1)^k a_k}\right)\cdot \frac{1}{2^n}\sum_{j=0}^{k}\binom{n}{j} \\
&= \sum_{k=0}^{n}(-1)^k a_k \left(1 - \frac{1}{2^n}\sum_{j=0}^{k}\binom{n}{j}\right) \\
&= \sum_{k=0}^{n}(-1)^k a_k \frac{1}{2^n}\left(2^n - \sum_{j=0}^{k}\binom{n}{j}\right) \\
&= \sum_{k=0}^{n}(-1)^k a_k \frac{1}{2^n}\left(\sum_{j=0}^{k}\binom{n}{j} - \sum_{j=0}^{k}\binom{n}{j}\right) \\
&= \sum_{k=0}^{n-1}(-1)^k a_k \frac{1}{2^n}\sum_{j=k+1}^{n}\binom{n}{j}
\end{aligned}$$

since $a_k \neq 0$ for all $k$. Since the expression is 0 for $k = n$, because

$$1 - \frac{1}{2^n}\sum_{j=0}^{k}\binom{k}{j} = 1 - \frac{1}{\sum_{j=0}^{n}\binom{n}{j}}\sum_{j=0}^{k}\binom{k}{j}.$$

So for $k = n$ we get that

$$1 - \frac{1}{\sum_{j=0}^{n}\binom{n}{j}}\sum_{j=0}^{n}\binom{n}{j} = 1 - 1 = 0.$$

We can write it as $\sum_{k=0}^{n}(-1)^k a_k$ as $\sum_{k=0}^{n-1}(-1)^k a_k$ in the above equation. This may look completely opaque but it's actually the $(n-1)$th partial sum of the Euler transform. We note that the $(n-1)$th partial sum of the Euler transform is

$$\sum_{j=0}^{n-1}\frac{1}{2^{j+1}}\sum_{k=0}^{j}(-1)^k\binom{n}{j}a_k = \sum_{k=0}^{n-1}(-1)^k a_k \sum_{j=k}^{n-1}\frac{1}{2^{j+1}}\binom{j}{k}.$$

Note that we are summing over the same set of couples.

$$M_n := \{(j,k) \in \mathbb{N}\times\mathbb{N} : 0 \leq k \leq j \leq n-1\}.$$

So for any double sequence $(A_{(j,k)})_{j,k}$ we have

$$\sum_{j=0}^{n-1}\sum_{k=0}^{j} A(j,k) = \sum_{(j,k)\in M_n} A(j,k) = \sum_{k=0}^{n-1}\sum_{j=k}^{n-1} A(j,k).$$

So now we need to prove that

$$\frac{1}{2^n}\sum_{j=k+1}^{n}\binom{n}{j} = \sum_{j=k}^{n-1}\frac{1}{2^{j+1}}\binom{j}{k}.$$

This is done by induction. We want to show that for every positive integer $n$, and for every integer $0 \le k < n$, we have

$$\frac{1}{2^n}\sum_{j=k+1}^{n}\binom{n}{j} = \sum_{j=k}^{n-1}\frac{1}{2^{j+1}}\binom{j}{k}.$$

When $n = 1$ we must have $k = 0$ so both sides equal $\frac{1}{2}$. Suppose that the above statement is true for some positive integer $n$ and for every $0 \le k < n$. We must show that for every $0 \le k \le n$, we have

$$\frac{1}{2^{n+1}}\sum_{j=k+1}^{n+1}\binom{n+1}{j} = \sum_{j=k}^{n}\frac{1}{2^{j+1}}\binom{j}{k}.$$

When $k = n$ both sides equal $\frac{1}{2^{n+1}}$, so we can assume $0 \le k < n$. We now have that

$$\begin{aligned}
\frac{1}{2^{n+1}}\sum_{j=k+1}^{n+1}\binom{n+1}{j} &= \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}\sum_{j=k+1}^{n}\binom{n+1}{j} \\
&= \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}\sum_{j=k+1}^{n}\left[\binom{n}{j} + \binom{n}{j-1}\right] \\
&= \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}\sum_{j=k+1}^{n}\binom{n}{j} + \frac{1}{2^{n+1}}\sum_{j=k+1}^{n}\binom{n}{j-1} \\
&= \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}\sum_{j=k+1}^{n}\binom{n}{j} + \frac{1}{2^{n+1}}\sum_{j=k}^{n-1}\binom{n}{j} \\
&= \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}\sum_{j=k+1}^{n}\binom{n}{j} + \frac{1}{2^{n+1}}\sum_{j=k}^{n}\binom{n}{j} + \frac{1}{2^{n+1}} \\
&= \frac{1}{2^{n+1}}\left(\sum_{j=k+1}^{n}\binom{n}{j}\cdot 2 + \binom{n}{k}\right) \\
&= \frac{1}{2^n}\sum_{j=k+1}^{n}\binom{n}{j} + \frac{1}{2^{n+1}}\binom{n}{k} \\
&= \sum_{j=k}^{n-1}\frac{1}{2^{j+1}}\binom{j}{k} + \frac{1}{2^{n+1}}\binom{n}{k} \\
&= \sum_{j=k}^{n}\frac{1}{2^{j+1}}\binom{j}{k}.
\end{aligned}$$

This is exactly what we needed to show. Hence, this completes the proof. $\square$

When we apply the Euler transformation on Leibniz's series we get a nice result $\pi = 2 \cdot (1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9} + ...)$ with a fast convergence rate. Let's give a formal proof of this.

**Theorem 4.** $\pi = 2 \cdot \sum\limits_{k=0}^{\infty} \frac{k!}{3\cdot 5\cdot \ldots \cdot (2k+1)}$ *with the computation-time* $O(2^{-n})$

*Proof.* We will apply the Euler transformation on Leibniz's series. We have that $a_k = \frac{1}{2k+1}$. and we will find the weight-function corresponding to this series. Simply chose the weight-function $w(t) = t^{-1/2} \cdot \frac{1}{2}$. Thereby we have

$$\frac{1}{2k+1} = \int_0^1 t^k \left(t^{-1/2} \cdot \frac{1}{2}\right) dt$$

This is a true weight-function since

$$\int_0^1 t^k \cdot w(t) dt = \int_0^1 t^k \left(t^{-1/2} \cdot \frac{1}{2}\right) dt = \frac{1}{2} \int_0^1 t^{k-0.5} dt$$

$$= \frac{1}{2} \left[\frac{1}{k+0.5} t^{k+0.5}\right]_0^1 = \frac{1}{2k+1}.$$

We also notice that this weight-function is unbounded on $[0,1]$, however, it has a finite integral over that interval. From the theorem 3 we know that the Euler transform of this series converges rapidly to its sum. We will now find the Euler transform of this series. We have that

$\nabla^0 a_0 = 1$

$\nabla^1 a_0 = 1 - \frac{1}{3} = \frac{2}{3}$

$\nabla^2 a_0 = 1 - 2 \cdot \frac{1}{3} + \frac{1}{5} = \frac{8}{15} = \frac{2\cdot 4}{3\cdot 5}$

$\nabla^3 a_0 = 1 - 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{5} - \frac{1}{7} = \frac{16}{35} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$.

From these indications, we assume that

$$\nabla^k a_0 = \frac{2 \cdot 4 \cdot \ldots \cdot 2k}{3 \cdot 5 \cdot \ldots \cdot (2k+1)}.$$

We can prove this by induction. By definition we have

$$\nabla^k a_0 = \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \binom{k}{j}.$$

Let us turn this to a polynomial function.

$$\xi_k(x) = \sum_{j=0}^{k} (-1)^j \frac{x^{2j+1}}{2j+1} \binom{k}{j},$$

so $\xi_k(1) = \nabla^k a_0$. Differentiating, we get

$$\frac{\partial \xi_k}{\partial x} = \sum_{j=0}^{k} (-1)^j \frac{1}{2j+1} \binom{k}{j} \frac{\partial (x^{2j+1})}{\partial x} = \sum_{j=0}^{k} (-x^2)^j \binom{k}{j} = (1-x^2)^k$$

where the last equality follows from the Binomial Theorem. Now since $\xi_k(0) = 0$ we get from integration by parts, recall that $\int_b^a u(x)v'(x)dx = [u(x)v(x)]_b^a - \int_b^a u'(x)v(x)dx$, the following

$$\begin{aligned}
\xi_k(1) &= \int_0^1 \frac{\partial \xi_k(t)}{\partial t} dt = \int_0^1 (1-t^2)^k dt \\
&= \int_0^1 (1-t^2) \cdot 1 dt = \big[t(1-t^2)\big]_0^1 + \int_0^1 2kt^2(1-t^2)^{k-1} dt \\
&= 0 + \int_0^1 2k(1-(1-t^2))(1-t^2)^{k-1} dt \\
&= 2k \int_0^1 1 \cdot (1-t^2)^{k-1} dt - 2k \int_0^1 (1-t^2)(1-t^2)^{k-1} dt \\
&= 2k\xi_{k-1}(1) - 2k\xi_k(1).
\end{aligned}$$

This is equivalent to $(2k+1)\xi_k(1) = 2k\xi_{k-1}(1)$, which is exactly the inductive step. Thus, we have

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \frac{2^k \cdot k!}{3 \cdot 5 \cdot \ldots \cdot (2k+1)} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{3 \cdot 5 \cdot \ldots \cdot (2k+1)}$$

and we know that from theorem 3 that the $n^{\mathrm{nh}}$ partial sum differs from $\frac{\pi}{4}$ by $O(2^{-n})$. This is a massive improvement of the convergence-rate for the Leibniz's series. Finally we have

$$\pi = 2 \cdot \sum_{k=0}^{\infty} \frac{k!}{3 \cdot 5 \cdot \ldots \cdot (2k+1)}$$

$$= 2 \cdot (1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7 \cdot 9} + \ldots).$$

□

## Further work and reading

Machin and Euler's are at best linear approximations, so neither one is really very good. The significance of Euler's work is not so much that it gives a very rapidly converging series but that it produces a very elegant one with at reasonable convergent-rate. Furthermore, the resulting transformed series can be used to produce a streaming algorithm that is quite easy to program and elegant in itself, and produces a large number of digits quite quickly and completely accurately.

In 1976 Richard Brent and Eugene Salamin independently and simultaneously derived a better method to compute digits of pi called the *Brent-Salamin algorithm*. The mathematics behind it is based on the work of the German mathematician Carl Friedrich Gauss (1777–1855) and the French mathematician Adrien-Marie Legendre (1752–1833) combined with modern algorithms for multiplication and square roots. The algorithm converges quadratically - each successive iteration roughly doubles the number of significant digits. All the other algorithms we have considered in this paper converge linearly - the number of significant digits is proportional to the number of iterations. The Brent-Salamin algorithm is rapidly convergent, with only 25 iterations producing 45 million correct digits of pi. but it's computer memory-intensive why Machin's approximation is still relevant. On September 18 to 20, 1999, it was used to compute the first 206,158,430,000 decimal digits of $\pi$[8].

## Calculating $\pi$ by programming

Let's consider how fast the terms are decreasing for both Euler's approach and Machin's approach.

$$\text{Euler's approach: } \left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{2n+1} \approx \frac{1}{2}.$$

$$\text{Machin's approach: } \left|\frac{u_{n+1}}{u_n}\right| = \frac{(2n+1)\cdot 5^{2n+1}}{(2n+3)\cdot 5^{2n+3}} = \frac{2n+1}{(2n+3)\cdot 25} \approx \frac{1}{25}.$$

We can estimate the error by the last term roughly. The error for Euler's approach is $\epsilon(n) \sim \frac{1}{2^n}$ and the error for Machin's approach is $\epsilon(n) \sim \frac{1}{25^n}$ for calculation $n$ terms. A program has been written in R to calculate the first digits of $\pi$ after only 10 iterations of both Euler's and Machin's approach.

### The R Code for Euler's approach

```
# Make a function with input m and out o_sum
# for Euler's approach
euler = function(m)  {
o=c()
for(k in 1:m) {
o[k]= prod(1:k)/ prod(2*(1:k)+1  )
}
o_sum=2*(1+sum(o)) # Final result
print(o_sum)
}
# Save all final results for Euler's approach
it = 10 # Number of iterations
saved_values_euler = c()
for(i in 1:it)  {
saved_values_euler[i] = euler(i)
}
sprintf("%.35f",saved_values_euler[1:it] )
```

The R code gives this output, where the correct digits of $\pi$ is highlighted with the orange colour.

[1]"2.66666666666666651863693004997912794"

[2]"2.93333333333333335701809119200333952 9"

[3]"3.04761904761904744987077719997614622"

[4]"3.09841269841269806306627287995070219"

[5]"3.12150072150072155352518166182562709"

[6]"3.13215673215673184870411205338314176"

[7]"3.13712953712953712681610340951010585"

[8]"3.13946968064615106186465709470212460"

[9]"3.14057816968033698401541187195107341"

[10]"3.14110602160137730720634863246232271"

### The R Code for Machin's approach

```
# Make a function for Machin's approach
machin=function(n) {
u=c()
v=c()
for(k in 1:n)  {
u[k]= 1/((2*k+1)*5^(2*k+1))  *(-1)^k
v[k]= 1/((2*k+1)*239^(2*k+1)) *(-1)^k
}
print(16*(sum(u)+1/5) - 4*(sum(v)+1/239))
}
n=10 # Number of iterations
saved_values_machin=c()
for(m in 1:n){
saved_values_machin[m]=machin(m)
}
sprintf("%.35f",(saved_values_machin)[1:n] )
```

The R code gives this output, where the correct digits of $\pi$ is highlighted in the orange colour.

[1]” 3.14 059702932606032987905564368702471”
[2]” 3.141 62102932503461971691649523563683”
[3]” 3.14159 177218217733340566155675332993”
[4]” 3.1415926 8240439981667577740154229105”
[5]” 3.14159265 261530862289873766712844372”
[6]” 3.141592653 62355499817681447893846780”
[7]” 3.14159265358 860251282635545067023486”
[8]” 3.141592653589 83619265131892461795360”
[9]” 3.14159265358979 178373033391835633665”
[10]” 3.141592653589793 56008717331860680133”

We see that in fact both approaches gives digits of $\pi$ already after the first iterations.

### Calculating more digits

To calculate many digits of $\pi$ we need to operate with both small and large numbers. We used the R package **Rmpfr - Multiple Precision Floating-Point Reliable**. We will use this to calculate thousands of digits of $\pi$. You can read more about the package on `https://cran.r-project.org/web/packages/Rmpfr/index.html`. We simply re-write the code in a syntax that keep the consistency.

```
# Create Euler's function
euler_fun = function(m, approx=TRUE){
  o = as.bigq(NULL)
  for(k in 1:m) {
    r = as.bigz(1:k)
    o = c(o, prod(r)/ prod(2*r+1 ))
  }
  o_sum = 2*(1+sum(o))
  if(approx){
    as.numeric(o_sum)
  }else{
    o_sum
  }
}

# Create Machin's function
```

```
mac = function(m, approx=TRUE){

      o1 = as.bigq(NULL)
          o2 = as.bigq(NULL)
          for(k in 1:m) {
k = as.bigq(k)
            o1 = c(o1, 1/((2*k+1)*5^(2*k+1))  *(-1)^k)
            o2 = c(o2, 1/((2*k+1)*239^(2*k+1))  *(-1)^k)
  }
  o_sum = 16*(1/as.bigz(5)+sum(o1)) -
     4*(sum(o2)+1/as.bigz(239))
  if(approx){
    as.numeric(o_sum)
  }else{
    o_sum
  }
}

# Using the Rmpfr-package for both functions
library(Rmpfr)
x = euler_fun(250, approx=FALSE)
mpfr(x,256) # Specify the number of bits
y = mac(250, approx=FALSE)
mpfr(y,256) # Specify the number of bits
```

Using Machin's function it took us less than a second to compute the first 100 digits of pi. Let's calculate the computation time.

```
system.time({
h=2000 # Choose a value for h
y = mac(h, approx=FALSE)
mpfr(y,h)
})
```

It took 36 seconds to compute the first 1,000 digits of $\pi$ with Machin's function. Simply increase the value for $h$ to get more digits. If we want to calculate millions of digits of $\pi$ the approach is not optimal, since we operate with very large and small numbers. For this reason one can consider Fourier Analysis for this purpose.

## Calculating $\pi$ by hand

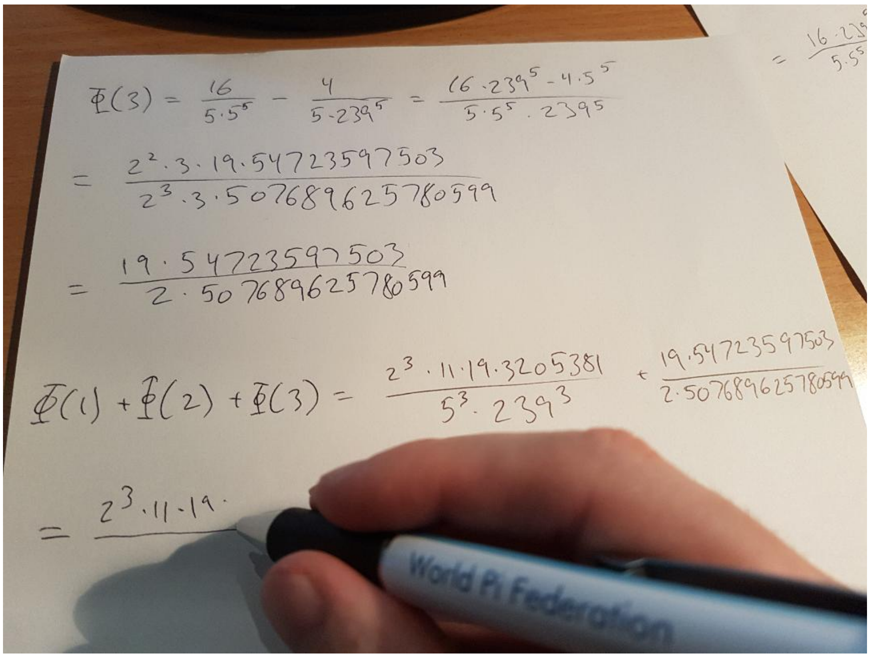
We will now calculate the first digits of $\pi$ by hand. The only thing we need is a pen and papers. This mean that we could perform the calculation if we were stranded on a deserted island. Here is a resume of the procedure. We calculated each iteration explicitly, founded the common numerator and founded some prime numbers in the factorization to reduce the expressions. We simply define the iteration function as $\Phi$, so $\pi \approx \sum_{i=1}^{m} \Phi(i)$ and the larger value of $m$ we have, the better results of $\pi$ we get.

$$\Phi(1) = \frac{16}{5} - \frac{4}{239} = \frac{16 \cdot 239 - 4 \cdot 5}{5 \cdot 239} = \frac{2^2 \cdot 3 \cdot 317}{5 \cdot 239}.$$

$$\Phi(2) = \frac{-16}{3 \cdot 5^3} + \frac{4}{3 \cdot 239^3} = \frac{-16 \cdot 239^3 + 4 \cdot 5^3}{3 \cdot 5^3 \cdot 239^3} = \frac{-2^2 \cdot 3 \cdot 18202517}{3 \cdot 5^3 \cdot 239^3} = \frac{-2^2 \cdot 18202517}{5^3 \cdot 239^3}.$$

$$\Phi(1) + \Phi(2) = \frac{2^2 \cdot 3 \cdot 317}{5 \cdot 239} + \frac{-2^2 \cdot 18202517}{5^3 \cdot 239^3} = \frac{2^2 \cdot 3 \cdot 317 \cdot 239^2 - 2^2 \cdot 18202517}{5^3 \cdot 239^3}$$

$$= \frac{2^3 \cdot 11 \cdot 19 \cdot 3205381}{5^3 \cdot 239^3}.$$




Picture of some of my calculations by hand

So, I thereby computed $\pi = 3.14159$. It took me **2 hours** to calculate the first **6 digits** of $\pi$. Now, six decimal places might not sound like a lot, but it's enough to accurately calculate the circumference of the Earth to within one meter. So, with an error of less than one meter, we can calculate the circumference of the Earth in about two hours by using a pen and some papers - nothing else.

## Conclusion

This paper has focused on two different approximations of $\pi$, namely, the approach of Euler and of Machin. It has demonstrated the mathematics in which they are based on. Furthermore, a program in R has been conducted that successfully calculate the first thousands digits of $\pi$. In fact it took about 36 seconds to calculate the first 1,000 digits of $\pi$. Some computations with pure hand power was demonstrated as well - it took me 2 hours to calculate the first 6 digits of $\pi$ by hand.

## References


[1]: http://www-groups.dcs.st-and.ac.uk/history/HistTopics/Pi_through_the_ages.html

[2]: https://www.businessinsider.com/archimedes-pi-estimation-2014-3?r=US&IR=T&IR=T

[3]: https://www.exploratorium.edu/pi/history-of-pi

[4]: http://worldpifederation.org/Fed/giza.pdf

[5]: https://mathsjam.com/assets/talks/2015/ColinBeveridge-GaussianMachinations.pdf

[6]: https://commons.wikimedia.org/wiki/File:Leonhard_Euler_2.jpg

[7]: Henri: Henri Cohen, Fernando Rodriguez Villegas, and Don Zagier, *Convergence Acceleration of Alternating Series* (2000).

[8]: http://www.cecm.sfu.ca/~jborwein/Kanada_200b.html