The decimal representation of π

By Grandmaster Mark Aarøe Nissen

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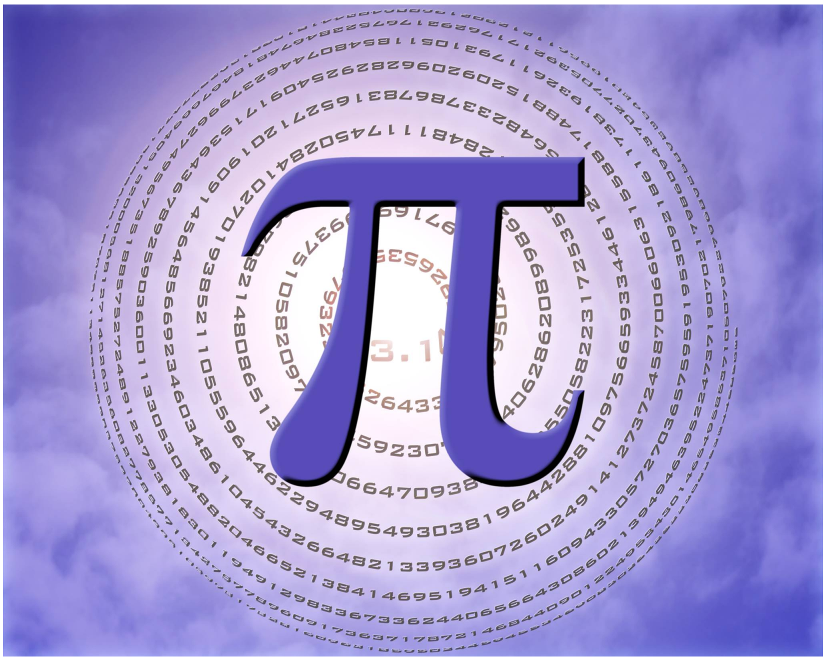

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Introduction

The goal of this paper is to investigate the decimal representation of π. We will see if the decimal representation is unique and if there exist repeating sequences. Finally, we will prove that π is not a rational number. Recall that a rational number $q = \frac{a}{b} \in \mathbb{Q}$ where $a, b \in \mathbb{Z}$ such that $b \neq 0$. Furthermore, we use the convention that the number 0 is included in the set of natural numbers.

History of irrational numbers

Mathematical problems involving irrational numbers such as $\sqrt{2}$, which is the diagonal in the unit square, were addressed very early during the Vedic period in India. The Indian scientist Manava (c. 750 – 690 BC) believed that the square roots of numbers such as 2,3,5 and so on, could not be exactly determined with rational numbers.[1]

The first mathematical proof of the existence of irrational numbers was carried out by a Pythagorean called Hippasus around 500 BC. He showed that $\sqrt{2}$ cannot be a rational number[2]. Greek mathematicians termed this ratio of incommensurable magnitudes *alogos*. Hippasus, however, was not lauded for his efforts: according to one legend, he made his discovery while out at sea, and was subsequently thrown overboard by his fellow Pythagoreans. Simply because he had discovered an element in the universe which denied the doctrine that all phenomena in the universe can be reduced to whole numbers and their ratios.[3]

There are other famous irrational numbers like π, Euler's number e, the golden ratio φ and many more. In fact as a consequence of Cantor's proof (1891) that real numbers are uncountable and the rational numbers countable, it follows that almost all real numbers are irrational.

In 1761, the German mathematician Johann Lambert proved that π is an irrational number. In the 19th century, Charles Hermite found a proof that requires no prerequisite knowledge beyond basic calculus. This proof has been simplified by Ivan Niven in 1947.[4] We will give a nice detailed proof of Ivan Niven's simplification by using basic calculus. Keep in mind that the proof shows us that π cannot be a rational number. We will not prove that π exists as a real number, however, given that π exists as a real number the proof shows us that π is an irrational number.

Definition 1

An **infinite decimal** is a sequence of the following form:

$$q.d_1d_2d_3...$$

where $q \in \mathbb{N}$ and d_i is a digit, ie a natural number between 0 and 9. A **terminating decimal** is a decimal

$$q.d_1d_2...d_n00...$$

In other words it "ends" and it clearly represents the rational number

$$q + \frac{d_1}{10} + \frac{d_2}{10^2} + ... + \frac{d_n}{10^n}.$$

Definition 2

Given a positive real number $r \in \mathbb{R}_+$ the infinite **decimal expansion** of r is defined as follows: q is chosen so that $q \leq r < q + 1$. The digits are then chosen by induction:

(i) d_1 is chosen so that:

$$q + \frac{d_1}{10} \leq r < q + \frac{d_1}{10} + \frac{1}{10}.$$

Since $q \leq r < q + 1$ we have that $d_1 \in \{0, ..., 9\}$.

(ii) Each d_{n+1} is chosen so that:

$$q + \frac{d_1}{10} + ... + \frac{d_{n+1}}{10^{n+1}} \leq r < q + \frac{d_1}{10} + ... + \frac{d_n}{10^n} + \frac{1}{10^{n+1}}$$

Since $q + \frac{d_1}{10} + ... + \frac{d_n}{10^n} \leq r < q + \frac{d_1}{10} + ... + \frac{d_n}{10^n} + \frac{1}{10^n}$ we have that $d_{n+1} \in \{0, ..., 9\}$. This gives a sequence of terminating decimals, which are rational numbers,

$$q,\ q.d_1,\ q.d_1d_2,\ q.d_1d_2d_3, \text{etc.}$$

that converges to r. Thus, $r = q.d_1d_2d_3....$

Example 1

$a)$ The decimal expansion of $\frac{1}{3}$ is 0.3333... because

$$0.333...3 < \frac{1}{3} < 0.333...4$$

$b)$ We can decimal expand $\sqrt{2}$ as far as we want to by squaring:

$1^2 = 1 < 2 < 4 = 2^2$, so $1 < \sqrt{2} < 2$ so $q = 1$.

$(1.4)^2 = 1.96 < 2 < 2.25 = (1.5)^2$, so $d_1 = 4$.

$(1.41)^2 = 1.9881 < 2 < 2.0164 = (1.42)^2$, so $d_2 = 1$.

$(1.414)^2 = 1.999396 < 2 < 2.002225 = (1.415)^2$, so $d_3 = 4$.

These are simply the first digits of the infinite decimal expansion of $\sqrt{2}$. It does not mean that there are any pattern unlike the infinite decimal expansion of $\frac{1}{3}$.

Definition 3

Given a positive rational number $\frac{l}{m}$, where $l, m \in \mathbb{N} : m \neq 0$. We perform the following divisions with remainders to define the digits of a decimal:

First we set $l = mq + r$, this defines q and an integer $r < m$.

Next, define the digits by induction:

(i) Set $10r = md_1 + r_1$, this defines d_1, which is a digit, and $r_1 < m$.

(ii) Set each $10r_n = md_{n+1} + r_{n+1}$, this defines d_{n+1}, which is a digit, as well as $r_{n+1} < m$,

and this defines digits d_n and remainders $r_n < m$ for all n by induction.

Proposition 1. *The infinite decimal in the rational expansion of $\frac{l}{m}$ is equal to its decimal expansion.*

Proof. Divide $l = mq + r$ by m to get:

$$(*)\frac{l}{m} = q + \frac{r}{m} \text{ and then } q \leq \frac{l}{m} < q + 1, \text{ since } 0 \leq \frac{r}{m} < 1,$$

so this is the correct q. Next a proof by induction checks the decimals:

(i) Divide $10R = md_1$ by $10m$ to get $\frac{r}{m} = \frac{d_1}{10} + \frac{r_1}{10m}$, and substitute into $(*)$ to get:

$$\frac{l}{m} = q + \frac{d_1}{10} + \frac{r_1}{10m}$$

which proves that d_1 is the correct digit, since $0 \leq \frac{r}{m} < 1$.

(ii) Once we know that $d_1, ..., d_n$ are the correct first n digits and that

$$(**)\frac{l}{m} = q.d_1d_2...d_n + \frac{r_n}{10^n m},$$

then divide $10r_n = md_{n+1} + r_{n+1}$ by $10^{n+1}m$ to get $\frac{r_n}{10^n m} = \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m}$, and substitute into $(**)$ to get:

$$\frac{l}{m} = q.d_1d_2...d_n + \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m} = q.d_1d_2...d_{n+1} + \frac{r_{n+1}}{10^{n+1}m}.$$

This proves that d_{n+1} is also correct, and completes the proof by induction. $\square$

Definition 4

A **repeating decimal** is any decimal of the form:

$$q.d_1d_2...d_kd_{k+1}...d_nd_{k+1}...d_nd_{k+1}...$$

for some pair of natural numbers $k < n$. The notation for this is

$$q.d_1d_2...d_k\overline{d_{k+1}...d_n}.$$

Example 2

An example of a number with a repeating sequence is the famous rational number $\frac{22}{7}$. The Greek mathematician Archimedes (287-212 BC) showed that $\pi < \frac{22}{7}$.

$$\frac{22}{7} = 3.\overline{142857} = 3.142857142857142857...$$

Compared to the true value of pi, namely $\pi = 3.141592...$, we see the values are very close and for practical reasons mankind has used $\pi \approx \frac{22}{7}$ for calculations involving π. Since $\frac{22}{7} = 3 + \frac{1}{7}$ we let $\xi = 0.\overline{142857}$ and we will now show that $\xi = \frac{1}{7}$. We have matching digits and by subtraction we get

$$10^6\xi - \xi = 142857.\overline{142857} - 0.\overline{142857} = 142857.$$

Every natural number can be represented by a unique product of prime numbers. Therefore

$$\xi = \frac{142857}{999999} = \frac{3^2 \cdot 11 \cdot 13 \cdot 111}{3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 111} = \frac{1}{7}.$$

Proposition 2. *All the decimal expansions of rational numbers repeat.*

Proof. Consider again step (ii) in the rational expansion of $\frac{l}{m}$ above:

$$(ii) 10r_n = md_{n+1} + r_{n+1}.$$

From this step it follows that if $r_k = r_n$ for some $k < n$, then:

$$d_{k+1} = d_{n+1} \text{ and } r_{k+1} = r_{n+1},$$

since $0 = 10r_n - 10r_k = md_{n+1} + r_{n+1} - md_{k+1} - r_{k+1} = m(d_{n+1} - d_{k+1}) + r_{n+1} - r_{k+1}$ and $m \neq 0$. Since $r_{k+1} = r_{n+1}$ we have that

$$d_{k+2} = d_{n+2} \text{ and } r_{k+2} = r_{n+2}$$

and so on and it follows by induction. Thus, when the remainder repeats for the first time, the decimal repeats. Note that all the remainders are between 0 and $m - 1$ so it must eventually repeat. So by the time we have done m divisions with remainders, we have to come across a repeat of the remainders. $\square$

Proposition 3. *Every repeating decimal is the decimal expansion of some rational number.*

Proof. Let a repeating decimal r be given so that $r = q.d_1 d_2 ... d_k \overline{d_{k+1} ... d_n}$. We have that

$$10^k r = (10^k q + 10^{k-1} d_1 + ... + d_k).\overline{d_{k+1} ... d_n}$$

and

$$10^n r = (10^n q + 10^{n-1} d_1 + ... + d_n).\overline{d_{k+1} ... d_n}.$$

These are matching decimals and from subtraction we get that

$$10^n r - 10^k r = (10^n q + ... + d_n) - (10^k q + ... + d_k)$$

and dividing both sides by $10^n - 10^k$, we see that r is rational:

$$r = \frac{(10^n q + ... + d_n) - (10^k q + ... + d_k)}{10^n - 10^k}.$$

□

As a consequence of Proposition 2 and Proposition 3 it follows that all irrational numbers do not have repeating decimals.

We will now investigate the uniqueness of decimal representations. There are numbers which have non-unique decimal representations. The example below shows us that $1 = 1.000... = 0.999...$, ie the number 1 has two different decimal representations.

Example 3

Let $\epsilon = 1.\overline{0} = 1.000...$ and $\delta = 0.\overline{9} = 0.999...$ and notice the following

$$10\delta = 9.\overline{9}.$$

We have matching decimals so by subtraction we get

$$9\delta = 10\delta - \delta = 9.\overline{9} - 0.\overline{9} = 9.$$

Ie $\delta = \frac{9}{9} = 1 = \epsilon$.

Proposition 4. *A number with a non-unique decimal representation will have exactly two representations: one ending in all* 0*s and one ending in all* 9*s.*

Proof. Take any positive number $a \in [0,1]$ with decimal representation

$$a = \sum_{n=1}^{\infty} \frac{a_n}{10^n},$$

where $a_n \in \{0, 1, ..., 9\}$. Suppose that

$$a = \sum_{n=1}^{\infty} \frac{b_n}{10^n}$$

for some $b_n \in \{0, 1, ..., 9\}$. Then

$$\sum_{n=1}^{\infty} \frac{a_n - b_n}{10^n} = 0.$$

Observe that $|a_n - b_n| \leq 9$, so that for any $N \in \mathbb{N}$, we use the basic sum rules for geometric series and we get that

$$|\sum_{n=N}^{\infty} \frac{a_n - b_n}{10^n}| \leq \sum_{n=N}^{\infty} \frac{9}{10^n} = 9 \cdot \sum_{n=0}^{\infty} (\frac{1}{10})^n - 9 \cdot \sum_{k=0}^{N-1} (\frac{1}{10})^n$$
$$= 9 \cdot \frac{1}{1 - \frac{1}{10}} - 9 \cdot (\frac{1 - (\frac{1}{10})^{N-1}}{1 - \frac{1}{10}}) = 10 - 10 + \frac{1}{10^{N-1}} = \frac{1}{10^{N-1}}$$

with equality if and only if $|a_n - b_n| = 9$ for each $n \in \mathbb{N}$. Now suppose that $(a_n) \neq (b_n)$ and let N be the first location where the sequences differ, so that

$$\sum_{n=1}^{\infty} \frac{a_n - b_n}{10^n} = \sum_{n=N}^{\infty} \frac{a_n - b_n}{10^n} = 0.$$

Then we have that

$$\frac{b_N - a_N}{10^N} = \sum_{n=N+1}^{\infty} \frac{a_n - b_n}{10^n}.$$

But

$$|\frac{b_N - a_N}{10^N}| \geq \frac{1}{10^N} \geq |\sum_{n=N+1}^{\infty} \frac{a_n - b_n}{10^n}|.$$

We know that these inequalities are really equalities, so without loss of generality we have that $a_n - b_n = 9$ for each $n > N$. Given the constraint $a_n, b_n \in \{0, 1, ..., 9\}$, this is only possible if $a_n = 9$ and $b_n = 0$. Therefore, a number with a non-unique decimal representation will have exactly two representations: one ending in all 0s and one ending in all 9s. $\square$

As a consequence it follows that all irrational numbers have an unique decimal representation.

Theorem. *Pi is not a rational number.*

Proof. We will prove the statement by contradiction. Assume that π is a rational number, ie there exist $a, b \in \mathbb{N} : b \neq 0$ where $\pi = \frac{a}{b}$ and we will find a contradiction. This will show us that π can not be a rational number.

Step 1

Lets fix a value $n \in \mathbb{N}$ so we can define the function

$$\alpha(x) = \frac{x^n(a-bx)^n}{n!}.$$

We will specify the value of n later.

Define a new function

$$\beta(x) := \alpha - \alpha^{(2)}(x) + \alpha^{(4)}(x) - ... + (-1)^n\alpha^{(2n)}(x).$$

Claim 1: $\beta(0) + \beta(\pi)$ is an integer.

Notice that by expanding out $(a - bx)^n$ we get

$$\begin{aligned}\alpha(x) &= \frac{x^n}{n!}(a_0 + a_1x + ... + a_nx^n)\\ &= \frac{a_0x^n}{n!} + \frac{a_1x^{n+1}}{n!} + ... + \frac{a_nx^{2n}}{n!}\end{aligned}$$

for some integer a_i where $i \in \{0, ..., n\}$.

Case 1: For $k < n$ we have

$$\begin{aligned}\alpha^{(k)}(x) = \frac{a_0n}{n!}x^{n-1} + ... + \frac{a_kn(n-1)\cdot ... \cdot(n-k)}{n!}x^{n-k}\\ +... \frac{a_n(2n)(2n-1)\cdot ... \cdot(2n-k)}{n!}x^{2n-k}.\end{aligned}$$

For $k < n$ we clearly see that $\alpha^{(k)}(0) = 0$. In other words we have differentiated too few times.

Case 2: For $k \geq n$ we get 0 if $k > 2n$. In other words we have differentiated too many times. Letting $k \in [n, 2n]$ we see that the only term that matters is the k term which is $\frac{a_k}{n!}x^k$. The kth derivative of this is

$$(\frac{a_k}{n!}x^k)^{(k)} = \frac{a_kk!}{n!}.$$

so $\alpha^{(k)}(0) = \frac{a_kk!}{n!}$. From our assumption $k \geq n$, so $\frac{k!}{n!}$ is an integer and therefore $\frac{a_k}{n!}k!$ is an integer. So $\alpha^{(k)}(0)$ is an integer for all k. From this it follows that $\beta(0)$ is also an integer.

We observe that

$\alpha(\pi - x) = (\pi - x)^n(a - b(\pi - x))^n/n!$

$= (\pi - x)^n(a - b\pi + bx)^n/n!$

$= (\pi - x)(a - a + bx)^n/n!$

$= (\frac{a}{b} - x)^n b^n x^n/n!$

$= ((\frac{a}{b} - x)b)^n x^n/n!$

$= \frac{x^n(a-bx)^n}{n!} = \alpha(x).$

Hence we have the derivatives

$\alpha(x) = \alpha(\pi - x)$

$\alpha(x)' = -\alpha(\pi - x)'$

$\alpha(x)'' = \alpha(\pi - x)'$

...

$\alpha(x)^{(k)} = (-1)^k\alpha(\pi - x)^{(k)}.$

So for $x = 0$ we get $\alpha^{(k)}(0) = (-1)^k\alpha^{(k)}(\pi)$ and $\alpha^{(k)}(\pi) = (-1)^k\alpha^k(0)$ is an integer, since $\alpha^k(0)$ is an integer for all k. Therefore $\beta(\pi)$ is an integer as well. Hence $\beta(0) + \beta(\pi)$ is an integer.

Step 2

Claim 2: $\beta(0) + \beta(\pi) = \int_0^\pi \alpha(x)\sin(x)dx$.

We notice that

$\beta(x)'' + \beta(x)$

$= (\alpha(x) - \alpha^{(2)}(x) + ... + (-1)^n\alpha^{(2n)}(x))''$

$+\alpha(x) - \alpha^{(2)}(x) + ... + (-1)^n\alpha^{(2n)}(x)$

$= \alpha(x)^{(2)} - \alpha^{(4)}(x) + ... + (-1)^n\alpha^{(2n+2)}(x)$

$+\alpha(x) - \alpha^{(2)}(x) + ... + (-1)^n\alpha^{(2n)}(x)$

$= \alpha(x)$

Note that $(-1)^n\alpha^{(2n+2)}(x) = 0$ because we differentiate a polynomial of degree $2n$ more than $2n$ times. So $\beta(x)'' + \beta(x) = \alpha(x)$. Hence

$$(\beta(x)'\sin(x) - \beta(x)\cos(x))'$$

$$= \beta(x)''\sin(x) + \beta(x)'\cos(x) - \beta(x)'\cos(x) + \beta(x)\sin(x)$$

$$= \beta(x)'' \sin(x) + \beta(x)\sin(x) = (\beta(x)'' + \beta(x))\sin(x) = \alpha(x)\sin(x).$$

From the Fundamental Theorem of Analysis it follows that

$$\int_0^\pi (\beta(x)' \sin(x) - \beta(x)\cos(x))' dx$$

$$= \beta(\pi)' \sin(\pi) - \beta(\pi)\cos(\pi) - \beta(0)' \sin(0) + \beta(0)\cos(0)$$

$$= \beta(\pi) + \beta(0) = \int_0^\pi \alpha(x)\sin(x)dx.$$

Hereby we have shown that claim 2 is true.

Step 3

So, we know that $\beta(\pi) + \beta(0) = \int_0^\pi \alpha(x)\sin(x)dx$ is an integer and we will now show that the integral has value in the interval strictly between 0 to 1 for all x.

Claim 3: $0 < \int_0^\pi \alpha(x)\sin(x)dx < 1$. From this it would follow that $0 < \beta(0) + \beta(\pi) < 1$, ie an integer is strictly between 0 and 1. Such an integer does not exist and thereby we have our contradiction.

Lets start to show its strictly above 0. Notice $x > 0$ since $x \in (0, \pi)$ and since a and b are strictly positive, we have

$$0 < x < \frac{a}{b} \Rightarrow bx < a \Rightarrow 0 < a - bx$$

so $\alpha(x) > 0$ and $\sin(x) > 0$ for all $x \in (0, \pi)$. Thereby we have shown that $0 < \int_0^\pi \alpha(x)\sin(x)dx$.

We will now show that this is strictly below 1.

Notice that when $x \in (0, \pi)$ we have that $\sin(x) \leq 1$ and

$$x(a - bx) = ax - bx^2 \leq ax < a\pi$$

so

$$\alpha(x) = x^n(a - bx)^n/n! < a^n\pi^n/n!.$$

Combining these inequalities it follows that

$$0 < \int_0^\pi \alpha(x)\sin(x)dx < \int_0^\pi \frac{(a\pi)^n}{n!}dx = \frac{(a\pi)^n}{n!}\int_0^\pi 1dx = \frac{(a\pi)^n\pi}{n!}.$$

Note for any $c, r > 0$ $\frac{c \cdot r^n}{n!} < 1$ for large enough n. This is a well known result, and you can prove it by induction if you like to.

From this is follows that $\frac{(a\pi)^n\pi}{n!} < 1$ for large enough n. Hereby we have specified our n and we have our contradiction. $\square$

Conclusion

As a consequence of Proposition 2, Proposition 3 and Proposition 4 it follows that all irrational numbers have a unique decimal representation and do not have repeating decimals. We proved that π is an irrational number. Therefore, we know that the decimal representation of π is unique and do not have repeating decimals.

References

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[2]: Kurt Von Fritz (1945). "The Discovery of Incommensurability by Hippasus of Metapontum". The Annals of Mathematics.

[3]: Kline 1990, p. 32.

[4]: `https://projecteuclid.org/euclid.bams/1183510788`